# Fast-forward of quantum adiabatic dynamics in electro-magnetic field

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We show a method to accelerate quantum adiabatic dynamics of wavefunctions under electro-magnetic field by developing the previous theory (Masuda & Nakamura 2008 and 2010). Firstly we investigate the orbital dynamics of a charged particle. We derive the driving field which accelerates quantum adiabatic dynamics in order to obtain the final adiabatic states except for the spatially uniform phase such as the adiabatic phase in any desired short time. Fast-forward of adiabatic squeezing and transport in the electro-magnetic field is exhibited. Secondly we investigate spin dynamics under the magnetic field, showing the fast-forward of adiabatic spin inversion and of adiabatic dynamics in Landau-Zener model. The connection of the present framework with Kato-Berry's transitionless quantum driving is elucidated in Appendix.

**Keywords: mechanical control of atoms, electro-magnetic field, spin dynamics**

## 1. Introduction

The technology to manipulate tiny objects is rapidly evolving, and nowadays we can control even individual atoms (Eigler & Schweizer 1990). And various methods to control quantum states have been reported in Bose Einstein condensates (BEC) (Leggett 2001; Ketterle 2002; Leanhardt *et al.* 2002; Gustavson *et.al.* 2002), in quantum computing with the use of spin states (Nielsen & Chuang 2000) and in many other fields of applied physics. It would be important to consider the acceleration of such manipulations of quantum states for manufacturing purposes and for innovation of technologies. In our previous paper (Masuda & Nakamura 2008), we proposed the acceleration of quantum dynamics with the use of the additional phase of wavefunctions (WFs). We can accelerate a given quantum dynamics and exactly obtain a target state in any desired short time, where the target state is defined as the final state in a given standard dynamics. This kind of acceleration is called fast-forward of quantum dynamics.

One of the most important application of the theory of fast-forward is the acceleration of quantum adiabatic dynamics (Masuda & Nakamura 2010). The adiabatic process occurs when the external parameter of Hamiltonian of the system is adiabatically changed. Quantum adiabatic theorem (Born & Fock 1928; Kato 1950; Messiah 1962) states that, if the system is initially in an eigenstate of the

instantaneous Hamiltonian, it remains so during the adiabatic process (Berry 1984; Aharonov & Anandan 1987; Samuel & Bhandari 1988; Shapere & Wilczek 1989; Nakamura & Rice 1994; Bouwmeester *et.al.* 1996; Farhi *et.al.* 2001; Roland & Cerf 2002; Sarandy & Lidar 2005). The rate of change in the parameter of Hamiltonian with respect to time is infinitesimal, so that it takes infinite time to reach the final state in the adiabatic process. However, by using our theory (Masuda & Nakamura 2010), the target states (final adiabatic states) are available in any desired short time. There, infinitesimally-slow change in the adiabatic dynamics is compensated by the infinitely-rapid fast-forward.

On the other hand, electro-magnetic field (EMF) is often used to control quantum states *e.g.*, in manifestation of the quantum Hall effect (Thouless *et.al.* 1982) and manipulation of Bose-Einstein Condensates (Ketterle 2002). The acceleration of the adiabatic dynamics in EMF is far from being trivial, and therefore it is highly desirable to extend our theory of the fast-forward to systems under EMF. In this paper, we extend our previous theory of the fast-forward to the system in EMF, and derive a driving field to generate the target adiabatic state exactly besides from a spatially uniform phase like dynamical and adiabatic phases (Berry 1984).

In section 2, we explain the method of the standard fast-forwarding under EMF. Section 3 is devoted to the fast-forward of the adiabatic dynamics with EMF. Examples of orbital dynamics of a charged particle are given in Section 4. Section 5 is concerned with spin dynamics in magnetic field (MF). Summary and discussion are given in Section 6. In Appendix, we shall elucidate the connection of the present framework with Kato-Berry's transitionless quantum driving.

## 2. Standard fast-forward

Before embarking on the fast-forward of adiabatic dynamics, we derive the driving electro-magnetic field (EMF) which accelerates the (non-adiabatic) standard dynamics of wavefunction (WF) and drive it from an initial state to the target state which is defined as the final state of the standard dynamics. Hamiltonian for the system with electric field $\mathbf{E}_0$ and MF $\mathbf{B}_0$ corresponding to vector potential $\mathbf{A}_0$ is written as $H_0 = \frac{1}{2m_0}(\mathbf{p} + \frac{e}{c}\mathbf{A}_0)^2$. The electric and magnetic field are related to $\mathbf{A}_0$ as $\mathbf{E}_0 = -\frac{d\mathbf{A}_0}{dt}$ and $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$, respectively. For simplicity of notation, we put $\frac{e}{c}$ to be 1 hereafter, and we consider in the main text the case where in *a priori* there is no scalar potential $V_0$. The results for systems with $V_0$ is shown in Appendix A. The Schrödinger equation with a nonlinearity constant $c_0$ (appearing in macroscopic quantum dynamics) is represented as

$$i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m_0}\nabla^2\Psi_0 - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_0)\Psi_0 - \frac{i\hbar}{m_0}\mathbf{A}_0\cdot\nabla\Psi_0 + \frac{\mathbf{A}_0^2}{2m_0}\Psi_0$$
$$-c_0|\Psi_0|^2\Psi_0. \tag{2.1}$$

$\Psi_0$ is a known function of space $\mathbf{x}$ and time $t$ and is called a standard state. For any long time $T$ called a standard final time, we choose $\Psi_0(t = T)$ as a target state that we are going to generate. Let $\Psi_\alpha(\mathbf{x}, t)$ be a virtually fast-forwarded state of $\Psi_0(\mathbf{x}, t)$ defined by

$$|\Psi_\alpha(t) >= |\Psi_0(\alpha t) >, \tag{2.2}$$

where $\alpha \, (> 0)$ is a time-independent magnification factor of the fast-forward.

In general, the magnification factor can be time-dependent. Hereafter $\alpha$ is assumed to be time-dependent, $\alpha = \alpha(t)$. In this case, the virtually fast-forwarded state is defined as,

$$|\Psi_\alpha(t) > = |\Psi_0(\Lambda(t)) >, \tag{2.3}$$

where

$$\Lambda(t) = \int_0^t \alpha(t')dt'. \tag{2.4}$$

Since the generation of $\Psi_\alpha$ requires an anomalous mass reduction, we can not generate $\Psi_\alpha$ as it stands (Masuda & Nakamura 2008). But we can obtain the target state by considering a fast-forwarded state $\Psi_{FF} = \Psi_{FF}(\mathbf{x}, t)$ which differs from $\Psi_\alpha$ by an additional space-dependent phase, $f = f(\mathbf{x}, t)$, as

$$\Psi_{FF}(t) = e^{if}\Psi_\alpha(t) = e^{if}\Psi_0(\Lambda(t)), \tag{2.5}$$

where $f = f(\mathbf{x}, t)$ is a real function of space $\mathbf{x}$ and time $t$ and is called the additional phase. Schrödinger equation for fast-forwarded state $\Psi_{FF}$ is supposed to be given as

$$
\begin{aligned}
i\hbar\frac{\partial\Psi_{FF}}{\partial t} \;=\; &-\frac{\hbar^2}{2m_0}\nabla^2\Psi_{FF} - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_{FF})\Psi_{FF} - \frac{i\hbar}{m_0}\mathbf{A}_{FF}\cdot\nabla\Psi_{FF} + \frac{\mathbf{A}_{FF}^2}{2m_0}\Psi_{FF} \\
&+ V_{FF}\Psi_{FF} - c_0|\Psi_{FF}|^2\Psi_{FF}.
\end{aligned}
\tag{2.6}
$$

$V_{FF} = V_{FF}(\mathbf{x}, t)$ and $\mathbf{A}_{FF} = \mathbf{A}_{FF}(\mathbf{x}, t)$ are called a driving scalar potential and a driving vector potential, respectively. Driving EMF is related with $V_{FF}$ and $\mathbf{A}_{FF}$ as

$$\mathbf{E}_{FF} \;=\; -\frac{d\mathbf{A}_{FF}}{dt} - \nabla V_{FF}, \tag{2.7}$$

$$\mathbf{B}_{FF} \;=\; \nabla \times \mathbf{A}_{FF}. \tag{2.8}$$

If we appropriately tune the time dependence of $\alpha$ (the detail will be shown later), the additional phase can vanish at the final time of the fast-forward $T_F$, and we can obtain the exact target state

$$\Psi_{FF}(T_F) = \Psi_0(T), \tag{2.9}$$

where $T_F$ is the final time of the fast-forward defined by

$$T = \int_0^{T_F} \alpha(t)dt. \tag{2.10}$$

Equation (2.6) is rewritten with the use of equations (2.1), (2.4) and (2.5) as

$$
\begin{aligned}
-\hbar\frac{\partial f}{\partial t}\Psi_0 \;+\; &\alpha\Big[-\frac{\hbar^2}{2m_0}\nabla^2\Psi_0 - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_0)\Psi_0 - \frac{i\hbar\mathbf{A}_0\cdot\nabla\Psi_0}{m_0} + \frac{\mathbf{A}_0^2}{2m_0}\Psi_0\Big] \\
&= -\frac{\hbar^2}{2m_0}[\nabla^2\Psi_0 + i(\nabla^2 f)\Psi_0 + 2i\nabla f\cdot\nabla\Psi_0 - (\nabla f)^2\Psi_0] \\
&\quad -\frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_{FF})\Psi_0 - \frac{i\hbar}{m_0}\mathbf{A}_{FF}\cdot\{i(\nabla f)\Psi_0 + \nabla\Psi_0\} + \frac{\mathbf{A}_{FF}^2}{2m_0}\Psi_0 \\
&\quad +V_{FF}\Psi_0 + (\alpha - 1)c_0|\Psi_0|^2\Psi_0, 
\end{aligned}
\tag{2.11}
$$

where $f(\mathbf{x},t)$, $\Psi_0(\mathbf{x},\Lambda(t))$, $\alpha(t)$, $\mathbf{A}_0(\mathbf{x},\Lambda(t))$, $\mathbf{A}_{FF}(\mathbf{x},t)$ and $V_{FF}(\mathbf{x},t)$ are abbreviated by $f$, $\Psi_0$, $\alpha$, $\mathbf{A}_0$, $\mathbf{A}_{FF}$ and $V_{FF}$, respectively. The same abbreviation will be used throughout in this section. Real and imaginary parts of equation (2.11) divided by $\Psi_0$ yield a pair of equations as

$$|\Psi_0|^2 \nabla \cdot (\nabla f - \frac{\alpha \mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar}) \quad + \quad 2\mathrm{Re}[\Psi_0 \nabla \Psi_0^*](\nabla f - \frac{\alpha \mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar})$$
$$+ \quad (\alpha - 1)\mathrm{Im}[\Psi_0 \nabla^2 \Psi_0^*] = 0, \qquad (2.12)$$

and

$$\frac{V_{FF}}{\hbar} = -\frac{\partial f}{\partial t} - (\alpha - 1)\frac{\hbar}{2m_0}\mathrm{Re}[\nabla^2 \Psi_0/\Psi_0]$$
$$-\frac{\hbar}{m_0}(\nabla f - \frac{\alpha \mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar})\mathrm{Im}[\nabla \Psi_0/\Psi_0] - \frac{\hbar}{2m_0}(\nabla f)^2 + \frac{\hbar}{2m_0}\frac{\alpha \mathbf{A}_0^2 - \mathbf{A}_{FF}^2}{\hbar^2}$$
$$-\frac{\hbar}{m_0}\frac{\mathbf{A}_{FF}}{\hbar} \cdot \nabla f - (\alpha - 1)\frac{c_0}{\hbar}|\Psi_0|^2. \qquad (2.13)$$

We can take the driving scalar potential from equation (2.13) and the additional phase $f$ which is a solution of equation (2.12).

### (a) *Additional phase and driving field*

In order to obtain the driving field, we should first calculate the additional phase in equation (2.12). Here we derive a general solution of equation (2.12) from the continuity equation for $\Psi_0$ and $\Psi_{FF}$. With the use of equation (2.1), we have a continuity equation for $\Psi_0$

$$\frac{\partial}{\partial t}|\Psi_0|^2 = \frac{\hbar}{m_0}\nabla \cdot (\mathrm{Im}[\Psi_0 \nabla \Psi_0^*] - \frac{\mathbf{A}_0}{\hbar}|\Psi_0|^2), \qquad (2.14)$$

and by using equations (2.5) and (2.6), the continuity equation for $\Psi_{FF}$ is

$$\frac{\partial}{\partial t}|\Psi_{FF}|^2 = \frac{\hbar}{m_0}\nabla \cdot (-\nabla f|\Psi_0|^2 + \mathrm{Im}[\Psi_0 \nabla \Psi_0^*] - \frac{\mathbf{A}_{FF}}{\hbar}|\Psi_0|^2), \qquad (2.15)$$

where $\Psi_{FF}(\mathbf{x},t)$ is abbreviated by $\Psi_{FF}$. From equation (2.5) which is the definition of $\Psi_{FF}$, we have a relation between time derivatives of $\Psi_0$ and $\Psi_{FF}$ as

$$\frac{\partial}{\partial t}|\Psi_{FF}|^2 = \alpha \frac{\partial}{\partial t}|\Psi_0|^2. \qquad (2.16)$$

Combining equations (2.14), (2.15) and (2.16), we have the gradient of the additional phase

$$\nabla f(\mathbf{x},t) = (\alpha - 1)\mathrm{Im}[\nabla \Psi_0/\Psi_0] + (\alpha \frac{\mathbf{A}_0}{\hbar} - \frac{\mathbf{A}_{FF}}{\hbar}). \qquad (2.17)$$

Noting the equivalence of gauges for $\mathbf{A}_0$ and $\mathbf{A}_{FF}$ due to the initial condition ($\mathbf{A}_0(t=0) = \mathbf{A}_{FF}(t=0)$), we can take the gradient of the additional phase and $\mathbf{A}_{FF}$ as

$$\nabla f = (\alpha - 1)\mathrm{Im}[\nabla \Psi_0/\Psi_0], \qquad (2.18)$$
$$\mathbf{A}_{FF} = \alpha \mathbf{A}_0. \qquad (2.19)$$

And it is easily confirmed that equations (2.18) and (2.19) satisfy equation (2.12). Equation (2.19) implies that MF should be magnified by $\alpha$ times, that is,

$$\mathbf{B}_{FF}(\mathbf{x}, t) = \alpha(t)\mathbf{B}_0(\mathbf{x}, \Lambda(t)), \tag{2.20}$$

where $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ and $\mathbf{B}_{FF} = \nabla \times \mathbf{A}_{FF}$. When the standard state $\Psi_0$ is written with its phase $\eta(\mathbf{x}, t)$ as

$$\Psi_0(\mathbf{x}, t) = |\Psi_0(\mathbf{x}, t)|e^{i\eta(\mathbf{x}, t)}, \tag{2.21}$$

equation (2.18) leads the expression of the additional phase and its gradient as

$$\nabla f(\mathbf{x}, t) = (\alpha - 1)\nabla\eta(\mathbf{x}, \Lambda(t)), \tag{2.22a}$$
$$f(\mathbf{x}, t) = (\alpha - 1)\eta(\mathbf{x}, \Lambda(t)). \tag{2.22b}$$

In equation (2.22b), a space-independent constant term was neglected.

Substitution of equation (2.19) into equation (2.13) yields the driving scalar potential

$$\begin{aligned}
\frac{V_{FF}}{\hbar} &= -\frac{\partial f}{\partial t} - (\alpha - 1)\frac{\hbar}{2m_0}\mathrm{Re}[\nabla^2\Psi_0/\Psi_0] \\
&\quad - \frac{\hbar}{m_0}\nabla f \cdot \mathrm{Im}[\nabla\Psi_0/\Psi_0] - \frac{\hbar}{2m_0}(\nabla f)^2 - \frac{\hbar}{m_0}\frac{\alpha\mathbf{A}_0}{\hbar} \cdot \nabla f \\
&\quad - \frac{\hbar}{2m_0}\alpha(\alpha - 1)\frac{\mathbf{A}_0^2}{\hbar^2} - (\alpha - 1)\frac{c_0}{\hbar}|\Psi_0|^2,
\end{aligned} \tag{2.23}$$

where $f$ and $\nabla f$ are given by equation (2.22). Therefore, once we have $\Psi_0$ and $\mathbf{A}_0$, the driving field can be obtained from equations (2.7) and (2.8) with the use of equation (2.19) and (2.23). By applying $\mathbf{E}_{FF}$ and $\mathbf{B}_{FF}$ against the initial standard state, we can generate the target state in any short time $T_F$ related to the standard final time through $\alpha$ by equation (2.10).

## 3. Fast-forward of adiabatic dynamics

So far, we presented the fast-forward of the standard dynamics in EMF which enables to generate the target state in any desired short time. Here we show the fast-forward of adiabatic dynamics of WF under EMF in an analogous manner which we employed in the previous paper (Masuda & Nakamura 2010). In the process, the regularized adiabatic dynamics is introduced as standard states and we fast-forward them with infinitely large magnification factor $\alpha$. In the acceleration, the initial state is stationary and it becomes back to stationary state at the end of the fast-forward. However we can not directly apply the theory in last section to adiabatic dynamics, because the stationary state is just an energy eigenstate of the instantaneous Hamiltonian and, therefore, is not suitable to be fast-forwarded. Thus we first need to regularize the adiabatic dynamics for the fast-forward.

Let us consider $\Psi_0$ under $\mathbf{E}_0$ and $\mathbf{B}_0$ corresponding to the vector potential $\mathbf{A}_0 = \mathbf{A}_0(\mathbf{x}, R(t))$ which adiabatically varies, where $R = R(t)$ is a parameter which is changed from constant $R_0$ as

$$R(t) = R_0 + \varepsilon t. \tag{3.1}$$

The constant value $\varepsilon$ is the rate of adiabatic change in $R(t)$ with respect to time and is infinitesimal, that is,

$$\frac{dR(t)}{dt} = \varepsilon, \tag{3.2}$$

$$\varepsilon \ll 1. \tag{3.3}$$

The EMF is changed through this parameter. Hamiltonian of the system is represented as

$$H_0 = \frac{(\mathbf{p} + \mathbf{A}_0(\mathbf{x}, R(t)))^2}{2m_0}, \tag{3.4}$$

and Schrödinger equation for $\Psi_0$ is given as

$$i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m_0}\nabla^2\Psi_0 - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_0)\Psi_0 - \frac{i\hbar}{m_0}\mathbf{A}_0\cdot\nabla\Psi_0 + \frac{\mathbf{A}_0^2}{2m_0}\Psi_0 - c_0|\Psi_0|^2\Psi_0, \tag{3.5}$$

where $c_0$ is a nonlinearity constant. The results for systems with $V_0$ are shown in Appendix A. If a system is in the $n$-th energy eigenstate at the initial time, the adiabatic theorem guarantees that, in the limit $\varepsilon \to 0$, $\Psi_0$ remains in the $n$-th energy eigenstate of the instantaneous Hamiltonian. Then $\Psi_0$ is written as

$$\Psi_0(\mathbf{x}, t, R(t)) = \phi_n(\mathbf{x}, R(t))e^{-\frac{i}{\hbar}\int_0^t E_n(R(t'))dt'}e^{i\Gamma(t)}, \tag{3.6}$$

where $E_n = E_n(R)$ and $\phi_n = \phi_n(\mathbf{x}, R)$ are the $n$-th energy eigenvalue and eigenstate corresponding to the parameter $R$, respectively, and $\Gamma = \Gamma(t)$ is the adiabatic phase given by

$$\Gamma(t) = i\int_0^t\int_{-\infty}^\infty d\mathbf{x}dt\phi_n^*\frac{d}{dt}\phi_n, \tag{3.7}$$

which is independent of space coordinates. $\phi_n$ fulfills

$$\frac{\partial\phi_n}{\partial t} = 0, \tag{3.8}$$

$$-\frac{\hbar^2}{2m_0}\nabla^2\phi_n - \frac{i\hbar}{2m_0}(\nabla\cdot\mathbf{A}_0)\phi_n \quad - \quad \frac{i\hbar}{m_0}\mathbf{A}_0\cdot\nabla\phi_n + \frac{\mathbf{A}_0^2}{2m_0}\phi_n - c_0|\phi_n|^2\phi_n = E_n\phi_n. \tag{3.9}$$

The second and the third factors of the right hand side of equation (3.6) are called dynamical and adiabatic phase factors, respectively, which are also space-independent. (We will not intend to realize these phase in the fast-forward.) The adiabatic dynamics in the limit $\varepsilon \to 0$ takes infinitely long time until we obtain an aimed adiabatic state (target state).

For the fast-forward of the adiabatic dynamics, we should first choose an appropriate standard state and Hamiltonian. The original adiabatic state is not appropriate as the standard state. Quantum dynamics in equation (3.5) with small but finite $\varepsilon$ inevitably induces non-adiabatic transition, but $\Psi_0$ in equation (3.6) ignores such transition. To overcome this difficulty, we regularize the standard state

and Hamiltonian corresponding to the adiabatic dynamics (Masuda & Nakamura 2010), so that the following two conditions are satisfied.

1. A regularized standard Hamiltonian and state of the fast-forward should agree with $H_0$ and $\Psi_0$ except for space-independent phase, respectively, in the limit $\varepsilon \to 0$;

2. The regularized standard state should satisfy the Schrödinger equation corresponding to the regularized standard Hamiltonian up to $O(\varepsilon)$ with small but finite $\varepsilon$.

Hereafter $\Psi_0^{(reg)}$ and $H_0^{(reg)}$ denote the regularized standard state and Hamiltonian, respectively, which fulfill the conditions 1 and 2. In the fast-forward, we take the limit $\varepsilon \to 0$, $\alpha \to \infty$ and $\alpha\varepsilon \sim 1$. Applying this regularization procedure in advance, the adiabatic dynamics $\phi_n(R(0)) \to \phi_n(R(T))$ can be accelerated and the target state $\phi_n(R(T))$ is realized in any desired short time, where $T$ is a standard final time which is taken to be $O(1/\varepsilon)$.

### (a) *Regularization of standard state*

Let us regularize the standard state so that it can be fast-forwarded with infinitely large magnification factor. Let us consider a regularized Hamiltonian $H_0^{(reg)}$

$$H_0^{(reg)} = \frac{(\mathbf{p} + \mathbf{A}^{(reg)})^2}{2m_0} + V_0^{(reg)}. \tag{3.10}$$

The scalar potential $V_0^{(reg)}$ in the regularized Hamiltonian is given as

$$V_0^{(reg)}(\mathbf{x}, t) = \varepsilon \tilde{V}(\mathbf{x}, t). \tag{3.11}$$

$\tilde{V}$ is a real function of $\mathbf{x}$ and $t$ to be determined *a posteriori*, which is introduced to incorporate the effect of non-adiabatic transitions. On the other hand, for the vector potential we put $\mathbf{A}^{(reg)}(\mathbf{x}, t) = \mathbf{A}_0(\mathbf{x}, t)$. It is obvious that $H_0^{(reg)}$ agrees with $H_0$ in the limit $\varepsilon \to 0$, that is,

$$\lim_{\varepsilon \to 0} H_0^{(reg)}(\mathbf{x}, t) = H_0(\mathbf{x}, R(t)). \tag{3.12}$$

The standard state in the adiabatic dynamics should fulfill Schrödinger equation up to $O(\varepsilon)$. We suppose that a regularized standard state is given by

$$\Psi_0^{(reg)} = \phi_n e^{-\frac{i}{\hbar}\int_0^t E_n(R(t'))dt'} e^{i\varepsilon\theta}, \tag{3.13}$$

where $\theta = \theta(\mathbf{x}, t)$ is real, and $\phi_n = \phi_n(\mathbf{x}, R(t))$ and $E_n = E_n(R(t))$ are the $n$-th energy eigenstate without dynamical phase factor and eigenvalue of the original Hamiltonian $H_0$. $\phi_n$ satisfies the instantaneous eigenvalue problem in equation (3.9). The Schrödinger equation for regularized standard system is represented as

$$\begin{aligned}
i\hbar\frac{\partial \Psi_0^{(reg)}}{\partial t} &= -\frac{\hbar^2}{2m_0}\nabla^2\Psi_0^{(reg)} - \frac{i\hbar}{2m_0}(\nabla \cdot \mathbf{A}_0)\Psi_0^{(reg)} - \frac{i\hbar}{m_0}\mathbf{A}_0 \cdot \nabla\Psi_0^{(reg)} + \frac{\mathbf{A}_0^2}{2m_0}\Psi_0^{(reg)} \\
&\quad + \varepsilon\tilde{V}\Psi_0^{(reg)} - c_0|\Psi_0^{(reg)}|^2\Psi_0^{(reg)}.
\end{aligned} \tag{3.14}$$

Substituting equation (3.13) into equation (3.14) and eliminating the equation of $O(1)$ with the use of equation (3.9), we find the equation for $O(\varepsilon)$:

$$i\hbar\frac{\partial \phi_n}{\partial R} - \hbar\frac{d\theta}{dt}\phi_n = -\frac{\hbar^2}{2m_0}[2i\nabla\theta \cdot \nabla\phi_n + i(\nabla^2\theta)\phi_n] + \tilde{V}\phi_n + \frac{\hbar}{m_0}\mathbf{A}_0 \cdot (\nabla\theta)\phi_n. \tag{3.15}$$

Multiplying equation (3.15) by $\frac{i}{\hbar}\phi_n^*$ and taking the the real and imaginary parts of the resultant equation, we have

$$|\phi_n|^2\nabla^2\theta + 2\text{Re}[\phi_n^*\nabla\phi_n]\cdot\nabla\theta + \frac{2m_0}{\hbar}\text{Re}[\phi_n^*\frac{\partial\phi_n}{\partial R}] = 0, \qquad (3.16)$$

$$\frac{\tilde{V}}{\hbar} = -\text{Im}[\frac{\partial\phi_n}{\partial R}/\phi_n] - \frac{\hbar}{m_0}\text{Im}[\nabla\phi_n/\phi_n]\cdot\nabla\theta - \frac{\hbar}{m_0}[\frac{\mathbf{A}_0}{\hbar}\cdot\nabla\theta]. \qquad (3.17)$$

From equation (3.16), $\theta$ turns out to be dependent on $t$ only through $R(t)$. Therefore the minor term $d\theta/dt(=\varepsilon\frac{\partial\theta}{\partial R})$ was suppressed in Eq(3.17). Equations (3.16) and (3.17) give $\theta$ and $\tilde{V}$, respectively. It is worth noting that $\theta$ is not explicitly affected by EMF.

(*b*) *Additional phase and driving field for fast-forward of adiabatic dynamics*

The regularized standard state in equation (3.13) is now written as

$$\Psi_0^{(reg)} = |\phi_n|e^{i(\eta+\varepsilon\theta)}e^{-\frac{i}{\hbar}\int_0^t E_n dt}, \qquad (3.18)$$

where $\eta$ is defined as a phase of $\phi_n$ by $\phi_n = |\phi_n|e^{i\eta}$. By using $\Psi_0^{(reg)}$ in equation (3.18) instead of $\Psi_0$ in equation (2.12), we have

$$|\phi_n|^2\nabla\cdot(\nabla f - \frac{\alpha\mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar}) + 2[|\phi_n|\nabla|\phi_n|](\nabla f - \frac{\alpha\mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar})$$
$$-(\alpha-1)[2|\phi_n|\nabla(\eta+\varepsilon\theta)\cdot\nabla|\phi_n| + |\phi_n|^2\nabla^2(\eta+\varepsilon\theta)] = 0, \qquad (3.19)$$

where $\phi_n(\mathbf{x}, R(\Lambda(t)))$, $f(\mathbf{x},t)$, $\mathbf{A}_0(\mathbf{x}, R(\Lambda(t)))$, $\mathbf{A}_{FF}(\mathbf{x},t)$, $\eta(\mathbf{x}, R(\Lambda(t)))$ and $\theta(\mathbf{x}, R(\Lambda(t)))$ are abbreviated by $\phi_n$, $f$, $\mathbf{A}_0$, $\mathbf{A}_{FF}$, $\eta$ and $\theta$ respectively, and the same abbreviations will be taken hereafter in this section.

Multiplying $\phi_n^*$ on both sides of equation (3.9) and taking its imaginary part with the use of $\phi_n = |\phi_n|e^{i\eta}$, we have

$$\frac{\hbar}{m_0}[2|\phi_n|\nabla\eta\cdot\nabla|\phi_n| + \nabla^2\eta|\phi_n|^2] + \frac{\hbar}{m_0}[\nabla\cdot\frac{\mathbf{A}_0}{\hbar}|\phi_n|^2 + 2|\phi_n|\nabla|\phi_n|\cdot\frac{\mathbf{A}_0}{\hbar}] = 0. \qquad (3.20)$$

With the use of equation (3.20) in equation (3.19), we obtain

$$|\phi_n|^2\nabla\cdot(\nabla f - \frac{\mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar}) + 2[|\phi_n|\nabla|\phi_n|](\nabla f - \frac{\mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar})$$
$$-(\alpha-1)\varepsilon[2|\phi_n|\nabla\theta\cdot\nabla|\phi_n| + |\phi_n|^2\nabla^2\theta] = 0. \qquad (3.21)$$

We can easily confirm that

$$\nabla f - \frac{\mathbf{A}_0 - \mathbf{A}_{FF}}{\hbar} = (\alpha-1)\varepsilon\nabla\theta \qquad (3.22)$$

satisfies equation (3.21). Noting $\mathbf{A}_{FF}(t=0) = \mathbf{A}_0(t=0)$, we have the vector potential $\mathbf{A}_{FF}$ and gradient of the additional phase from equation (3.22) as

$$\mathbf{A}_{FF}(t) = \mathbf{A}_0(\Lambda(t)) \qquad (3.23)$$
$$\nabla f = (\alpha-1)\varepsilon\nabla\theta, \qquad (3.24)$$

which should be compared with the result in equations (3.8) and (3.9) in the case of the standard fast-forward. It is noteworthy that we do not have to magnify the MF for the fast-forward, while in the standard fast-forward we need to magnify the MF by $\alpha$ times as shown in section 2. The result in equation (3.22) is also obtained from the continuity equation.

As mentioned in Section 2, in the standard fast-forward with a standard scalar potential $V_0$, the driving scalar potential is given by equation (A 1). By using $\Psi_0^{(reg)}$ in equation (3.13) and $V_0^{(reg)}$ in equation (3.11) instead of $\Psi_0$ and $V_0$ and noting equations (3.9), (3.17), (3.23) and (3.24), equation (A 1) leads to the driving scalar potential as

$$
\begin{aligned}
\frac{V_{FF}}{\hbar} \quad = \quad & (\alpha - 1)\frac{E_n}{\hbar} - \frac{d\alpha}{dt}\varepsilon\theta - \alpha^2\varepsilon^2\frac{\partial\theta}{\partial R} - \frac{\hbar}{2m_0}\alpha^2\varepsilon^2(\nabla\theta)^2 - \frac{\hbar}{m_0}\alpha\varepsilon\frac{\mathbf{A}_0}{\hbar}\cdot\nabla\theta \\
& -\alpha\varepsilon\mathrm{Im}[\frac{\partial\phi_n}{\partial R}/\phi_n] - \alpha\varepsilon\frac{\hbar}{m_0}\mathrm{Im}[\frac{\nabla\phi_n}{\phi_n}]\cdot\nabla\theta,
\end{aligned} \tag{3.25}
$$

where we omitted a term of $O(\varepsilon)$. While the first term diverges with infinitely large $\alpha$, it concerns only with spatially uniform phase of WF, which we do not care about in the fast-forward and can be omitted. Consequently, we have the driving scalar potential

$$
\begin{aligned}
\frac{V_{FF}}{\hbar} \quad = \quad & -\frac{d\alpha}{dt}\varepsilon\theta - \alpha^2\varepsilon^2\frac{\partial\theta}{\partial R} - \frac{\hbar}{2m_0}\alpha^2\varepsilon^2(\nabla\theta)^2 - \frac{\hbar}{m_0}\alpha\varepsilon\frac{\mathbf{A}_0}{\hbar}\cdot\nabla\theta \\
& -\alpha\varepsilon\mathrm{Im}[\frac{\partial\phi_n}{\partial R}/\phi_n] - \alpha\varepsilon\frac{\hbar}{m_0}\mathrm{Im}[\frac{\nabla\phi_n}{\phi_n}]\cdot\nabla\theta.
\end{aligned} \tag{3.26}
$$

The driving field can be obtained from equations (2.8), (3.23) and (3.26). So far we considered the fast-forward in the systems without scalar potential $V_0$. The driving scalar potential for systems with $V_0$ is shown in Appendix A, while the driving vector potential has the same form as in systems without $V_0$.

The present theory of the fast-forward is different from the reverse engineering approach based on the inverse technique (*e.g.*, Palao *et.al.* 1998) in the following sense: the latter approach is concerned with a direct fast-forward of the adiabatic state itself. By contrast, we consider the adiabatic states except for the spatially uniform phase but together with a controllable additional phase, and combine an idea of the infinitely-fast-forward and infinitesimally-slow adiabatic dynamics. Muga *et.al.* employed the reverse engineering approach to accelerate the adiabatic squeezing or expanding of BEC wave packet (Muga *et.al.* 2009; Chen *et.al.* 2010) in a tunable harmonic trap, finding a promising time dependence of the trapping frequency. While their approach is limited to WF under the harmonic trap, the present theory of the fast-forward enables to accelerate adiabatic dynamics of WF in any potential and EMF.

## 4. Examples

So far we showed theoretical framework of the fast-forward of adiabatic dynamics in EMF. Here we give some examples of the fast-forward of adiabatic dynamics without nonlinearity constant ($c_0 = 0$) for simplicity. Our purpose is the realization of target states defined in the adiabatic process in any desired short time, while

the target states are reached through infinitely long time in the original adiabatic dynamics. In the following examples, the magnification factor is commonly chosen (for $0 \le t \le T_F$) in the form

$$\alpha(t)\varepsilon = \bar{v}(1 - \cos(\frac{2\pi}{T_F}t)), \tag{4.1}$$

where $\bar{v}$ is time average of $\alpha(t)\varepsilon$ during the fast-forwarding, and the final time of the fast-forward $T_F$ is related to the standard final time $T$ with $\bar{v}$ as $T_F = \varepsilon T/\bar{v}$ (see equation (2.10)). $\varepsilon T$ and $T_F$ are taken as any finite value, although $\varepsilon$ is infinitesimal and $T$ is infinitely-large. Namely we aim to generate the target state in finite time, while the state is supposed to be obtained after infinitely long time $T$ in the original adiabatic dynamics. $\alpha\varepsilon$ starts from zero and becomes back to *zero* at the end of the fast-forward.

### (a) Fast-forward of adiabatic squeezing of wave packet with electro-magnetic field

We consider an adiabatically squeezed wave packet (WP) in two dimensions under the adiabatically increasing MF

$$\mathbf{B}_0 = (0, 0, R(t)), \tag{4.2}$$

where $R(t) = R_0 + \varepsilon t$ as given in equation (3.1). The vector potential corresponding to the MF can be taken as

$$\mathbf{A}_0 = (-R(t)y, 0, 0). \tag{4.3}$$

The lowest energy eigenstate $\phi_{n=0}$ of the instantaneous Hamiltonian with energy $E_0 = \frac{\hbar R}{2m_0}$ is given by

$$\phi_0 = \sqrt{\frac{R(t)}{2\pi\hbar}} e^{-\frac{R(t)}{4\hbar}(x^2+y^2)} e^{i\frac{R(t)xy}{2\hbar}}. \tag{4.4}$$

The corresponding regularized standard state is written with the phase of $O(\varepsilon)$ as

$$\Psi_0^{(reg)}(x,y,t) = \phi_0 e^{i\varepsilon\theta} e^{-\frac{i}{\hbar}\int_0^t E_0(R)dt} = \sqrt{\frac{R(t)}{2\pi\hbar}} e^{-\frac{R(t)}{4\hbar}(x^2+y^2)} e^{i\frac{R(t)xy}{2\hbar}} e^{-i\frac{1}{2m_0}\int_0^t R(t)dt} e^{i\varepsilon\theta}. \tag{4.5}$$

This WF is squeezed when MF is increased. We accelerate the manipulation which control the width of WP. Equation (3.16) for $\theta$ is rewritten as

$$\nabla^2\theta - \frac{R}{\hbar}x\frac{\partial\theta}{\partial x} - \frac{R}{\hbar}y\frac{\partial\theta}{\partial y} - \frac{m_0}{2\hbar^2}(x^2+y^2) + \frac{m_0}{\hbar R} = 0. \tag{4.6}$$

It can be easily confirmed that

$$\theta = -\frac{m_0}{4\hbar R}(x^2+y^2) \tag{4.7}$$

satisfies equation (4.6). Equation (3.17) leads to the regularized standard potential as

$$V_0^{(reg)} = \varepsilon\tilde{V} = -\varepsilon\frac{xy}{2}. \tag{4.8}$$

With the use of equations (4.5) and (4.7) in equation (3.26), the driving scalar potential is obtained as

$$\frac{V_{FF}}{\hbar} = [\frac{d\alpha}{dt}\varepsilon\frac{m_0}{4\hbar R} - \alpha^2\varepsilon^2\frac{3m_0}{8\hbar R^2}](x^2 + y^2) - \alpha\varepsilon\frac{xy}{2\hbar}, \tag{4.9}$$

where we omitted a spatially uniform term because it is concerns only with the spatially uniform phase. The driving EMF can be obtained from equations (2.7) and (2.8) with the use of equations (3.23), (4.9) and (4.3). For numerical calculation the parameters are chosen as $\frac{m_0}{\hbar} = 1.0, T_F = 1.0, \bar{v} = 1.0, R_0 = 1.0$. The WP profile is shown in figure 1 at the initial (upper figure) and final (lower figure) time of the fast-forward. It can be seen the WP is squeezed successfully. To check the accuracy
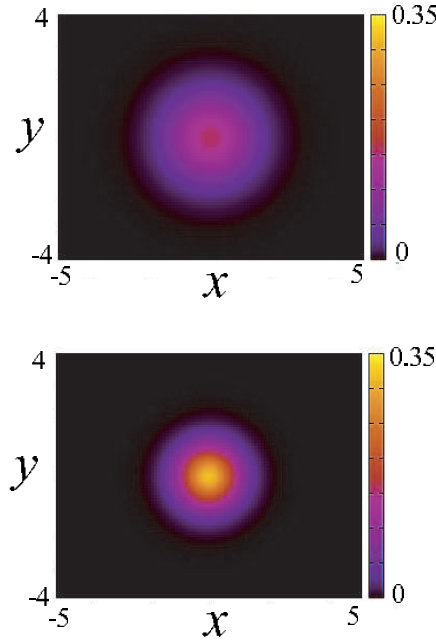


Figure 1. Wavefunction $|\Psi_{FF}|$ profile before (upper figure) and after (lower figure) the squeezing.

of the acceleration, we evaluated the fidelity which is defined by

$$F = | < \Psi_{FF}(t)|\Psi_0(\Lambda(t)) > |, \tag{4.10}$$

i.e., the overlap between the fast-forwarded state $\Psi_{FF}(t)$ and the corresponding standard one $\Psi_0(\Lambda(t))$. It is unity when $\Psi_{FF}(t) = \Psi_0(\Lambda(t))$. We confirmed that the fidelity first decreases from unity due to the additional phase $f$ of the fast-forwarded state, but at the final time it becomes unity again (see figure 2), which means the exact fast-forward of the adiabatic state besides from the spatially uniform phase factor.
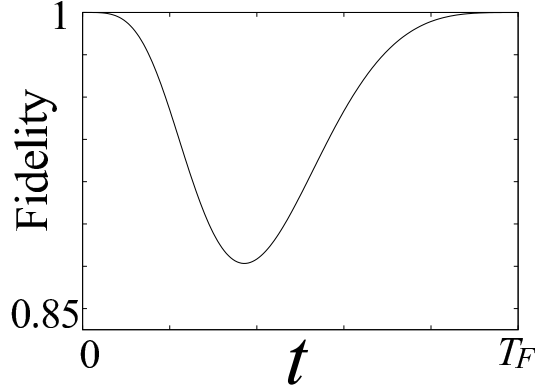
Figure 2. Time dependence of fidelity.

### (b) *Fast-forward of adiabatic transport in electro-magnetic field*

Here we show the fast-forward of adiabatic transport of WF in 2 dimensions subjected to EMF, without leaving any disturbance on the WF at the end of the transport.

WF takes a form $\psi(\mathbf{x})e^{-\frac{i}{\hbar}E_n t}$, which is stationary except for the adiabatic phase, in the presence of vector potential $\mathbf{A}(x, y, z)$ at the initial time. The MF is adiabatically shifted with infinitesimal velocity $\varepsilon$ in $x$-direction. The shifted vector potential $\mathbf{A}_0(\mathbf{x}, t)$ are represented with the use of $\mathbf{A}$ as

$$\mathbf{A}_0 = \mathbf{A}(x - \varepsilon t, y, z). \tag{4.11}$$

The corresponding regularized WF is supposed to be given in the form as

$$\Psi_0^{(reg)} = \phi_n e^{-\frac{i}{\hbar}E_n t} e^{i\varepsilon\theta}. \tag{4.12}$$

$\phi_n$ is the $n$-th energy eigenstate of the instantaneous Hamiltonian written as

$$\phi_n \quad = \quad \psi(x - \varepsilon t, y, z) = \psi(x - R(t), y, z). \tag{4.13}$$

$R(t)$ which characterizes the position of WP in $x$ direction is adiabatically changed as $R(t) = R_0 + \varepsilon t$ with $\varepsilon \ll 1$ and $R_0 = 0$. Equation (4.13) leads to the relation

$$\frac{\partial \phi_n}{\partial R} = -\frac{\partial \phi_n}{\partial x}. \tag{4.14}$$

In the same manner as used in the previous example, we can obtain $\theta$ and $\tilde{V}$ as

$$\theta \quad = \quad \frac{m_0}{\hbar}x,$$
$$\tilde{V} \quad = \quad -A_x. \tag{4.15}$$

Therefore the regularized potential $V_0^{(reg)}(x, y, z, t)$ in equation (3.11) is given as

$$V_0^{(reg)} = -\varepsilon A_x. \tag{4.16}$$

With the use of equations (4.14) and (4.15) in equation (3.26), we have the driving scalar potential

$$V_{FF}(x, y, z, t) = -m_0 \frac{d\alpha}{dt} \varepsilon x - \alpha \varepsilon A_x, \qquad (4.17)$$

where we omitted the spatially uniform term because it concerns only on the spatially uniform phase of WF. The driving MF is shifted with time and the corresponding vector potential is given from equation (3.23) as

$$\mathbf{A}_{FF}(x, y, z, t) = \mathbf{A}(x - R(\Lambda(t)), y, z). \qquad (4.18)$$

The driving electric field is given by

$$\mathbf{E}_{FF}(x, y, z, t) = \varepsilon \alpha \left( \frac{\partial \mathbf{A}_{FF}}{\partial R} \right) - \nabla V_{FF}. \qquad (4.19)$$

It is worth noting that since we have derived $\theta$ without giving any specific profile on $\phi_n$, the formulas of the driving scalar potential in equation (4.17) and driving vector potential in equation (4.18) are independent of the profile of the WF that we are going to transport. The resultant electric field due to this term and the time derivative of $\mathbf{A}_{FF}$ can be interpreted as balancing with Lorenz force perpendicular to the transport.

As a concrete example of such accelerated transport, we consider a case that a WP trapped in uniform MF $\mathbf{B}_0 = (0, 0, B)$. We choose the vector potential as

$$\mathbf{A}_0 = (0, B(x - \varepsilon t), 0), \qquad (4.20)$$

which leads to $\mathbf{B}_0$ and electric field of $O(\varepsilon)$ in $y$-direction:

$$\mathbf{E}_0 = -\frac{d\mathbf{A}_0}{dt} = (0, \varepsilon B, 0). \qquad (4.21)$$

In this case, $V_0^{(reg)} = 0$ as seen from equation (4.16). The WP is adiabatically moved due to $\theta$ and $\mathbf{E}_0$, while the MF does not change. From equations (4.18) and (4.20), it is obvious that we do not have to change MF for the fast-forward. An eigenstate with energy $E_0 = \frac{\hbar B}{2m_0}$ of the instantaneous Hamiltonian with $R$ is given as

$$\phi_n(x, y, t) e^{-i \frac{Bt}{2m_0}}, \qquad (4.22)$$

with

$$\phi_n(x, y, t) = \sqrt{\frac{B}{2\pi\hbar}} e^{-\frac{B}{4\hbar}((x - R(t))^2 + y^2)} e^{-i \frac{B(x - R(t))y}{2\hbar}}. \qquad (4.23)$$

Note that $\phi_n$ in equation (4.23) is a stationary state with an instantaneous value of $R$. We transport this state by the driving field. In this case, the driving scalar potential in equation (4.17) is represented as

$$V_{FF}(x, y, z, t) = -m_0 \frac{d\alpha}{dt} \varepsilon x, \qquad (4.24)$$

which leads to the driving electric field in $x$-direction. The driving electric field in $y$-direction given by

$$E_{FF}^{(y)} = -\frac{dA_{FF}^{(y)}}{dt} = \alpha\varepsilon B \qquad (4.25)$$

is balancing with Lorenz force in classical picture. Without this term the path of WP would be bent in $y$-direction.

In the numerical calculation the parameter are chosen as $\frac{m_0}{\hbar} = 1.0$, $T_F = 1.0$, $\bar{v} = 8.0$ and $R_0 = 0$. By applying the driving potential in equation (4.24), we accelerate WP. In figure 3, the WP profile $|\Psi_{FF}|$ is shown at the initial and final time of the fast-forward. The WP is transported by distance 8.0 in time 1.0 and becomes stationary at the end. We confirmed that WP is moved without changing its amplitude profile during the acceleration. We evaluated the fidelity defined by
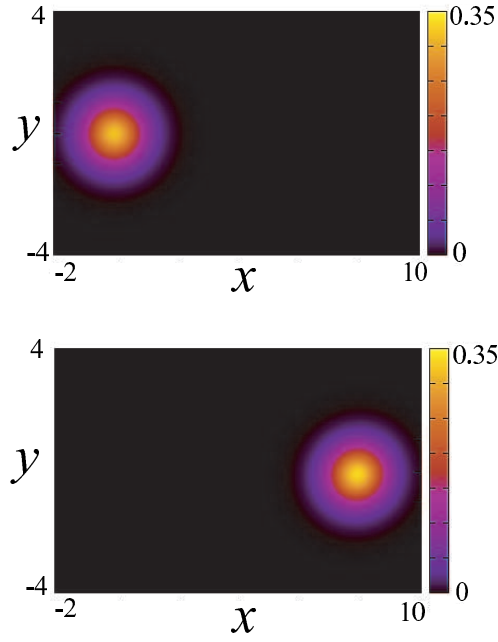


Figure 3. Wave packet profile $|\Psi_{FF}|$ at initial and final time of the fast-forward.

equation (4.10) and confirmed that it becomes back to unity at the end of the fast-forward (see figure 4). Thus we have obtained the adiabatically accessible target state in a finite time $T_F = 1.0$. The inset in figure 4 shows the path which the WP traces under the correct driving field (thick line) and the path without electric field in $y$-direction $\alpha\varepsilon B$ (thin line). Without this electric field, the orbit of WP is bent in $y$-direction by MF.

Here we considered the case that WP is trapped by MF. In Appendix A the driving field is shown in the case that WP is trapped by both MF and a scalar potential $V_0$.
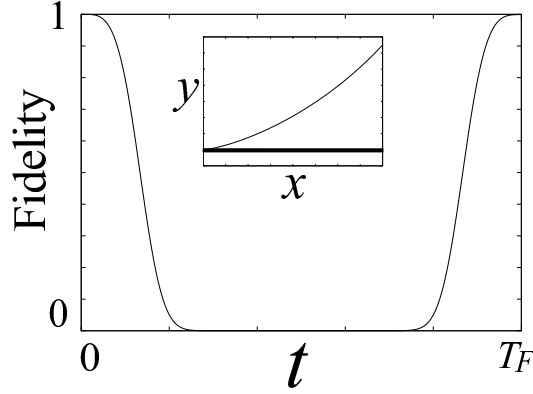
Figure 4. Time dependence of fidelity. The inset represents the path which the wave packet traces under the correct driving field (thick line) and one without electric field in $y$-direction (thin line).

## 5. Fast-forward of spin dynamics in magnetic field

We now proceed to the dynamics of spin with $S = \frac{1}{2}$, by suppressing the orbital degree of freedom. The fast-forward of spin dynamics in MF is as follows. Let us consider the Hamiltonian given by

$$H_0(t) = \frac{1}{2}(B_x^{(0)}(t)\sigma_x + B_y^{(0)}(t)\sigma_y + B_z^{(0)}(t)\sigma_z), \tag{5.1}$$

where $B_x^{(0)}$, $B_y^{(0)}$ and $B_z^{(0)}$ denote components of the MF and $\sigma_x$, $\sigma_y$, $\sigma_z$ are Pauri matrices. In equation (5.1) the negative of Bohr magneton was suppressed for simplicity. The MF can change with time $t$. Schrödinger equation can be given in $\sigma_z$-diagonal representation as

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & -H_{11} \end{pmatrix}\begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \end{pmatrix}, \tag{5.2}$$

where

$$H_{11}(t) = \frac{1}{2}B_z^{(0)}(t) \tag{5.3a}$$

$$H_{12}(t) = \frac{1}{2}(B_x^{(0)}(t) - iB_y^{(0)}(t)). \tag{5.3b}$$

The state

$$\Psi_0(t) = \begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \end{pmatrix} \tag{5.4}$$

is defined as a standard state that we are going to fast-forward. The exactly fast-forwarded state $\Psi_\alpha(t)$ is defined by $\Psi_0(\Lambda(t))$ with $\Lambda(t)$ in equation (2.4). Let the Hamiltonian for $\Psi_\alpha$ to be represented as

$$H_\alpha = \begin{pmatrix} H_{11}^{(\alpha)} & H_{12}^{(\alpha)} \\ (H_{12}^{(\alpha)})^* & -H_{11}^{(\alpha)} \end{pmatrix}. \tag{5.5}$$

The definition of the exactly fast-forwarded state in equation (2.3) leads to the relation

$$\frac{d\Psi_\alpha}{dt}\bigg|_{t=t'} = \alpha(t')\frac{d\Psi_0}{dt}\bigg|_{t=\Lambda(t')}. \tag{5.6}$$

With the use of Schrödinger equation for $\Psi_\alpha$ with Hamiltonian $H_\alpha$ together with equations (5.2) and (5.6), we have the relation between $H_\alpha$ and $H_0$ as

$$H_\alpha(t) = \alpha(t)H_0(\Lambda(t)). \tag{5.7}$$

Therefore we can obtain the driving MF, $\mathbf{B}_{FF}$, as

$$\mathbf{B}_{FF}(t) = \begin{pmatrix} B_x^{FF}(t) \\ B_y^{FF}(t) \\ B_z^{FF}(t) \end{pmatrix} = \begin{pmatrix} \alpha(t)B_x^{(0)}(\Lambda(t)) \\ \alpha(t)B_y^{(0)}(\Lambda(t)) \\ \alpha(t)B_z^{(0)}(\Lambda(t)) \end{pmatrix}. \tag{5.8}$$

In the fast-forward of spatially distributing WF, there was a problem of the anomalous mass reduction. To resolve this problem we introduced an additional phase $f$ on the fast-forwarded state (see equation (2.5)), but here in spin dynamics we do not have to use it. And we just require to magnify the MF to generate the exactly fast-forwarded state $\Psi_\alpha$.

### (a) Fast forward of adiabatic spin dynamics

Now, we show the fast-forward of adiabatic spin dynamics. Suppose that Hamiltonian which is adiabatically changed is represented as

$$H(R(t)) = \begin{pmatrix} H_{11}(R(t)) & H_{12}(R(t)) \\ H_{12}^*(R(t)) & -H_{11}(R(t)) \end{pmatrix}, \tag{5.9}$$

where $R(t)$ is the parameter which is adiabatically changed as in equation (3.1), namely, $\varepsilon \ll 1$. The matrix elements of the Hamiltonian are related to those of MF as,

$$H_{11}(R(t)) = \frac{1}{2}B_z^{(0)}(R(t)) \tag{5.10a}$$

$$H_{12}(R(t)) = \frac{1}{2}(B_x^{(0)}(R(t)) - iB_y^{(0)}(R(t))). \tag{5.10b}$$

Suppose

$$\Psi_0(R(t)) = \begin{pmatrix} c_1(R(t)) \\ c_2(R(t)) \end{pmatrix} e^{-\frac{i}{\hbar}\int_0^t E(R(t))dt}e^{i\Gamma(t)}, \tag{5.11}$$

to be an adiabatically evolving state. $\Gamma(t)$ is an adiabatic phase, which is common to both component. We have

$$E(R)\begin{pmatrix} c_1(R) \\ c_2(R) \end{pmatrix} = H(R)\begin{pmatrix} c_1(R) \\ c_2(R) \end{pmatrix}, \tag{5.12}$$

for an instantaneous Hamiltonian with a parameter $R$. In this fast-forward, we can not utilize the same manner as used in the standard fast-forward of spin dynamics

because the MF diverges due to infinitely large $\alpha$ $(= O(1/\varepsilon))$. Thus we need to regularize the standard Hamiltonian.

As we did in the fast-forward of adiabatic orbital dynamics, we regularize the system so that the Schrödinger equation is fulfilled up to $O(\varepsilon)$ for regularized standard state and Hamiltonian. In this case, however, there is no spatial distribution of WF, and therefore we do not have to regularize WF with any additional phase of $O(\varepsilon)$ corresponding to $\theta$ in equation (3.13). We simply put a regularized standard state without adiabatic phase $\Gamma$ as

$$\Psi_0^{(reg)}(R(t)) = \begin{pmatrix} c_1(R(t)) \\ c_2(R(t)) \end{pmatrix} e^{-\frac{i}{\hbar}\int_0^t E(R(t))dt}, \tag{5.13}$$

for which the regularized Hamiltonian is given by

$$H_0^{(reg)}(R(t)) = \begin{pmatrix} H_{11}(R(t)) + \varepsilon h_{11}(R(t)) & H_{12}(R(t)) + \varepsilon h_{12}(R(t)) \\ H_{12}^*(R(t)) + \varepsilon h_{12}^*(R(t)) & -H_{11}(R(t)) - \varepsilon h_{11}(R(t)) \end{pmatrix}, \tag{5.14}$$

introducing additional terms $h_{11}$ and $h_{12}$. Schrödinger equation is written as

$$i\hbar\frac{d\Psi_0^{(reg)}}{dt} = H_0^{(reg)}\Psi_0^{(reg)}. \tag{5.15}$$

The use of equation (5.12) in equation (5.15) leads

$$i\hbar\frac{\partial c_1}{\partial R} = h_{11}c_1 + h_{12}c_2, \tag{5.16a}$$

$$i\hbar\frac{\partial c_2}{\partial R} = h_{12}^*c_1 - h_{11}c_2. \tag{5.16b}$$

From equations (5.16) we can obtain

$$h_{11} = i\hbar(c_1^*\frac{\partial c_1}{\partial R} + c_2\frac{\partial c_2^*}{\partial R}), \tag{5.17a}$$

$$h_{12} = i\hbar(c_2^*\frac{\partial c_1}{\partial R} - c_1\frac{\partial c_2^*}{\partial R}), \tag{5.17b}$$

where $h_{11}$ in equation (5.17a) is purely real because $\frac{\partial}{\partial R}(|c_1|^2 + |c_2|^2) = 0$.

The fast-forwarded state is given as

$$\Psi_{FF}(t) = \begin{pmatrix} c_1(R(\Lambda(t))) \\ c_2(R(\Lambda(t))) \end{pmatrix} e^{-\frac{i}{\hbar}\int_0^t E(R(\Lambda(t)))dt}. \tag{5.18}$$

Since there is no additional phase on $\Psi_{FF}$, it is the exactly fast-forwarded state except for the adiabatic phase common to both components. We suppose that Hamiltonian $H_{FF}$ drives $\Psi_{FF}$. The time derivative of equation (5.18) is given by

$$\frac{d\Psi_{FF}}{dt} = \left(\alpha\varepsilon\begin{pmatrix} \frac{\partial c_1}{\partial R} \\ \frac{\partial c_2}{\partial R} \end{pmatrix} - \frac{i}{\hbar}E\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right)e^{-\frac{i}{\hbar}\int_0^t Edt}, \tag{5.19}$$

where $\Psi_{FF}(t)$, $\alpha(t)$, $E(R(\Lambda(t)))$, $c_1(R(\Lambda(t)))$ and $c_2(R(\Lambda(t)))$ are abbreviated by $\Psi_{FF}$, $\alpha$, $E$, $c_1$ and $c_2$, respectively, and the same abbreviations are used hereafter in this section. Schrödinger equation:

$$i\hbar\frac{d\Psi_{FF}}{dt} = H_{FF}\Psi_{FF} \tag{5.20}$$

and equations (5.12), (5.16) and (5.19) lead to the driving Hamiltonian as

$$
\begin{aligned}
H_{FF}(t) &= \frac{1}{2}\begin{pmatrix} B_z^{(FF)} & B_x^{(FF)} - iB_y^{(FF)} \\ B_x^{(FF)} + iB_y^{(FF)} & -B_z^{(FF)} \end{pmatrix} \\
&= \begin{pmatrix} H_{11} + \alpha\varepsilon h_{11} & H_{12} + \alpha\varepsilon h_{12} \\ H_{12}^* + \alpha\varepsilon h_{12}^* & -H_{11} - \alpha\varepsilon h_{11} \end{pmatrix}
\end{aligned}
\tag{5.21}
$$

where $H_{11}(R(\Lambda(t)))$, $H_{12}(R(\Lambda(t)))$, $h_{11}(R(\Lambda(t)))$ and $h_{12}(R(\Lambda(t)))$ are abbreviated by $H_{11}$, $H_{12}$, $h_{11}$ and $h_{12}$, respectively. The driving MF is obtained from equations (5.10), (5.17) and (5.21) as

$$
B_x^{(FF)} = B_x^{(0)} - 2\hbar\varepsilon\alpha\mathrm{Im}(c_2^*\frac{\partial c_1}{\partial R} - c_1\frac{\partial c_2^*}{\partial R}),
\tag{5.22a}
$$

$$
B_y^{(FF)} = B_y^{(0)} - 2\hbar\varepsilon\alpha\mathrm{Re}(c_2^*\frac{\partial c_1}{\partial R} - c_1\frac{\partial c_2^*}{\partial R}),
\tag{5.22b}
$$

$$
B_z^{(FF)} = B_z^{(0)} - 2\hbar\varepsilon\alpha\mathrm{Im}(c_1^*\frac{\partial c_1}{\partial R} + c_2\frac{\partial c_2^*}{\partial R}),
\tag{5.22c}
$$

where $B_i^{(FF)}(t)$ and $B_i^{(0)}(R(\Lambda(t)))$ are abbreviated by $B_i^{(FF)}$ and $B_i^{(0)}$, respectively, and $i$ denotes $x, y, z$. Noting $\varepsilon\alpha = O(1)$, the excess field, *i.e.*, the difference between $\mathbf{B}_{FF}$ and $\mathbf{B}_0$ gives a nontrivial contribution. By applying $\mathbf{B}_{FF} = (B_x^{(FF)}, B_y^{(FF)}, B_z^{(FF)})$ given in equation (5.22), we can accelerate adiabatic dynamics and obtain, in any short time, the exact target state except for the common phase between both components. It is obvious that $\Psi_{FF}$ and $\mathbf{B}_{FF}$ coincide with $\Psi_0$ and $\mathbf{B}_0$, respectively, at the initial and final time.

In closing this subsection, it should be emphasized: Our purpose in this paper lies in the fast-forward of the adiabatic dynamics except for the uniform phase. Therefore we suppressed the adiabatic phase in equation (5.13). Equation (5.22) is a result of such a simplified procedure. On the other hand, we can also accelerate the adiabatic dynamics with the adiabatic phase being included in equation (5.13). The resultant driving field is slightly different from equation (5.22), which is described in Appendix B. There, the driving field proved to be equal to the one obtained recently by Berry (Berry 2009).

### (b) Examples

As an example we consider the adiabatic dynamics in which MF is rotated adiabatically into opposite direction while its magnitude is kept constant. Let the MF be written as

$$
\mathbf{B} = B\mathbf{e}_r = B\begin{pmatrix} \sin\theta(t)\cos\varphi \\ \sin\theta(t)\sin\varphi \\ \cos\theta(t) \end{pmatrix},
\tag{5.23}
$$

where

$$
\theta(t) = R(t).
\tag{5.24}
$$

$R(t)$ is given by equation (3.1) with $R_0 = 0$. $B$ and $\varphi$ remain constant. $\mathbf{e}_r$ is a unit vector pointing the direction of MF. Hamiltonian for this MF is represented as

$$H(R(t)) \quad = \quad \frac{B}{2} \begin{pmatrix} \cos(R(t)) & \sin(R(t))e^{-i\varphi} \\ \sin(R(t))e^{i\varphi} & -\cos(R(t)) \end{pmatrix}. \tag{5.25}$$

Under this magnetic field, the regularized standard state corresponding to an adiabatic state with eigenvalue $\lambda_+ = \frac{B}{2}$ is given by

$$\Psi_0^{(reg)} \quad = \quad \begin{pmatrix} \cos\frac{R(t)}{2} \\ e^{i\varphi}\sin\frac{R(t)}{2} \end{pmatrix} e^{-\frac{iB}{2\hbar}t}. \tag{5.26}$$

From equations (5.17), (5.24) and (5.26), we have

$$h_{11} \quad = \quad 0, \tag{5.27a}$$

$$h_{12} \quad = \quad -\frac{\hbar}{2}(\sin\varphi + i\cos\varphi). \tag{5.27b}$$

From equations (5.21), (5.22), and (5.27), we can obtain the driving MF as

$$B_x^{(FF)}(t) \quad = \quad B\sin(R(\Lambda(t)))\cos\varphi - \varepsilon\alpha(t)\hbar\sin\varphi, \tag{5.28a}$$

$$B_y^{(FF)}(t) \quad = \quad B\sin(R(\Lambda(t)))\sin\varphi + \varepsilon\alpha(t)\hbar\cos\varphi, \tag{5.28b}$$

$$B_z^{(FF)}(t) \quad = \quad B\cos(R(\Lambda(t))), \tag{5.28c}$$

where $\Lambda(t)$ is given by equation (2.4). In numerical calculation the parameters are chosen as $\bar{v} = \pi$, $T_F = 1.0$ $\varphi = 0.0$, $R_0 = 0.0$, $R(T_F) = \pi$ and $B = 1.0$. Therefore in the fast-forward, the direction of MF switched into opposite direction in time $T_F$. Initially the state is set as

$$\Psi_{FF}(t = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.29}$$

Spin is rotated by the applied $\mathbf{B}_{FF}$ and points the opposite direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at the end of the fast-forward. In figure 5, the dynamics of the spin state: $|c_1^{(FF)}(t)|^2$ and $|c_2^{(FF)}(t)|^2$ are shown. It is confirmed that spin is flipped exactly from up to down at the final time $T_F$. The spin state becomes stationary after the acceleration again. We also confirmed that the fidelity defined between $\Psi_{FF}(t)$ and $\Psi_0(\Lambda(t))$ is unity during the fast-forward because there is no additional phase on the fast-forwarded state. We can control the direction of spin more generally, in any desired short time, by making $\varphi$ changing through $R$ like $\theta$.

As another example, we show the fast-forward of adiabatic dynamics in Landau-Zener (LZ) model (Landau 1932; Zener 1932). We consider MF:

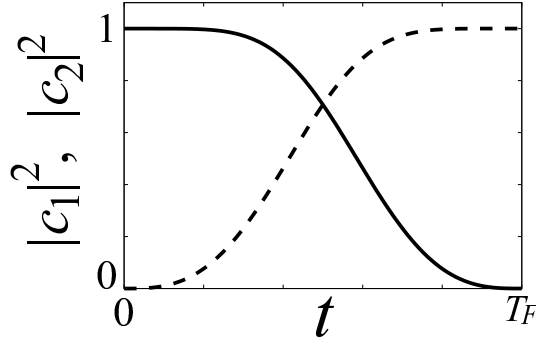$$\mathbf{B}(t) = \begin{pmatrix} \Delta \\ 0 \\ R(t) \end{pmatrix}, \tag{5.30}$$

Figure 5. The time dependence of $|c_1^{(FF)}|^2$ (solid line) and $|c_2^{(FF)}|^2$ (broken line).

where $\Delta$ is a constant, and $R(t)$ is given in equation (3.1) with large negative constant $R_0$. The Hamiltonian is given by

$$H(R(t)) = \frac{1}{2}\begin{pmatrix} R(t) & \Delta \\ \Delta & -R(t) \end{pmatrix}.$$ (5.31)

The adiabatic state with eigenvalue $\lambda_+ = \frac{\sqrt{R^2+\Delta^2}}{2}$ is given by

$$\Psi_0^{(reg)}(t) = \begin{pmatrix} -\Delta/s \\ \frac{R-\sqrt{R^2+\Delta^2}}{s} \end{pmatrix} e^{-\frac{i}{\hbar}\frac{\sqrt{R^2+\Delta^2}}{2}t},$$ (5.32)

where

$$s \equiv \{2\sqrt{R^2+\Delta^2}(\sqrt{R^2+\Delta^2}-R)\}^{1/2}.$$ (5.33)

Here we have

$$\frac{\partial c_1}{\partial R} = -\frac{1}{2\sqrt{2}}\frac{\Delta}{Q^{5/2}}(Q-R)^{\frac{1}{2}}$$ (5.34a)

$$\frac{\partial c_2}{\partial R} = \frac{1}{2\sqrt{2}}\frac{(Q-R)^{1/2}(Q+R)}{Q^{5/2}},$$ (5.34b)

where

$$Q \equiv \sqrt{R^2+\Delta^2}.$$ (5.35)

By using equations (5.32) and (5.34) in equation (5.22), we have the driving field as

$$\mathbf{B}_{FF}(t) = \begin{pmatrix} \Delta \\ -\varepsilon\alpha(t)\hbar\frac{\Delta}{R(\Lambda(t))^2+\Delta^2} \\ R(\Lambda(t)) \end{pmatrix},$$ (5.36)

The $y$-component of $\mathbf{B}_{FF}$ is identical to the one appearing in Landau-Majorana-Zener model (Landau 1932; Majorana 1932; Zener 1932; Berry 2009). The dynamics of the spin state driven by $\mathbf{B}_{FF}$ is shown in figure 6 during the fast-forward. The parameters were chosen as $\bar{v} = 100.0$, $T_F = 1.0$, $R_0 = -50.0$ and $\Delta = 1.0$. We confirmed the fidelity is unity throughout the acceleration. Interestingly, with the use of the formula different from equation (5.22), Berry also obtained equation (5.36). This mystery is solved in Appendix B.
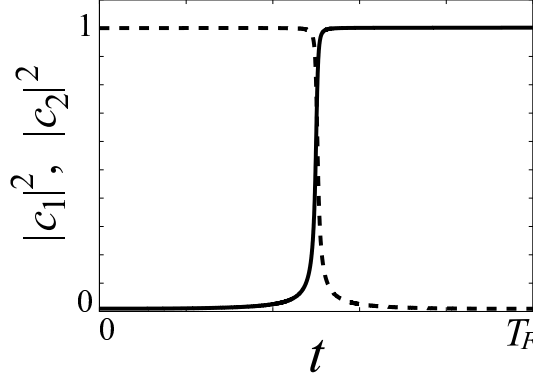
Figure 6. The time dependence of $|c_1^{(FF)}|^2$ (solid line) and $|c_2^{(FF)}|^2$ (broken line).

## 6. Conclusion

We have presented the theory of the fast-forward of quantum adiabatic dynamics in electro-magnetic field (EMF). We derived the driving EMF which accelerates the adiabatic dynamics and enables to obtain the final adiabatic states besides from the spatially uniform phase in any desired short time, while the final state is accessible after infinite time in the adiabatic dynamics. In the acceleration (fast-forward), the initial state is stationary and it becomes back to the stationary state at the end of the fast-forward without leaving any disturbance on the WF. For the fast-forward of adiabatic orbital dynamics of a charged particle, we must control the driving filed, but we do not have to magnify the magnetic field from that of adiabatic dynamics, while in the standard fast-forward, the magnification of the magnetic field is inevitable. As typical examples, we showed fast-forward of adiabatic wave packet squeezing by magnetic field and adiabatic transport in EMF. Furthermore we showed the fast-forward of adiabatic spin dynamics in time-dependent magnetic field. The distinction between the present theory and Kato-Berry's transitionless quantum driving was elucidated.

## Appendix A. Driving scalar potential in systems with scalar potential $V_0$

Here the driving scalar potential of the fast-forward of the systems with potential $V_0$ is shown. The driving vector potential is given by the same form as in the case without $V_0$.

For the fast-forward of regular (non-adiabatic) dynamics with the Hamiltonian $H_0 = \frac{1}{2m_0}(\mathbf{p} + \frac{e}{c}\mathbf{A}_0)^2 + V_0$, we have the driving scalar potential $\tilde{V}_{FF}$ as

$$\tilde{V}_{FF}(\mathbf{x}, t) = \alpha(t)V_0(\mathbf{x}, \Lambda(t)) + V_{FF}^{(1)}(\mathbf{x}, t), \tag{A 1}$$

where $V_{FF}^{(1)}$ is defined by $V_{FF}$ in equation (2.23). The driving vector potential is given by the same form as in equation (2.20).

The driving scalar potential for the fast-forward of adiabatic dynamics with Hamiltonian

$$H_0 = \frac{(\mathbf{p} + \mathbf{A}_0(\mathbf{x}, R(t)))^2}{2m_0} + V_0(\mathbf{x}, R(t)), \tag{A 2}$$

is given by

$$\tilde{V}_{FF}(\mathbf{x}, t) = V_0(\mathbf{x}, \Lambda(t)) + V_{FF}^{(2)}(\mathbf{x}, t), \tag{A 3}$$

where $V_{FF}^{(2)}$ is defined by $V_{FF}$ in equation (3.26).

In section 4, we showed the fast-forward of adiabatic transport of wave packet trapped by magnetic field. Here we show the driving scalar potential for adiabatic transport of wave packet trapped by electric and magnetic field with $V_0$. In adiabatic dynamics, the trapping scalar potential is also shifted as

$$V_0 = U(x - \varepsilon t, y, z), \tag{A 4}$$

as well as vector potential, where $U$ is a static trapping scalar potential. In such case, the driving scalar potential is represented as

$$\tilde{V}_{FF}(\mathbf{x}, t) = U(x - R(\Lambda(t)), y, z) + V_{FF}^{(3)}(\mathbf{x}, t), \tag{A 5}$$

where $V_{FF}^{(3)}$ is defined by $V_{FF}$ in equation (4.17).

## Appendix B. Fast-forward of adiabatic spin dynamics with adiabatic phase

We compare the driving field in equation (5.22) with the one obtained by Berry (Berry 2009) with the use of Kato's formalism (Kato 1950). Under the magnetic field

$$\mathbf{B}_0(R(t)) = B_0 \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}, \tag{B 1}$$

$c_1$ and $c_2$ in equation (5.11) are represented as

$$c_1 = \cos\frac{\theta}{2} \tag{B 2a}$$

$$c_2 = e^{i\varphi}\sin\frac{\theta}{2}, \tag{B 2b}$$

respectively. Here all of $\mathbf{B}_0$, $\theta$ and $\varphi$ are dependent on $R(t)$ which is slowly changing in time. Substituting equation (B 2) into equation (5.22), our driving field is expressed as

$$B_x^{(FF)}(t) = B_x^{(0)}(R) - 2\hbar\varepsilon\alpha(\frac{\partial}{\partial R}\frac{\theta}{2}\sin\varphi + \frac{\partial}{\partial R}\varphi\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\varphi), \tag{B 3a}$$

$$B_y^{(FF)}(t) = B_y^{(0)}(R) + 2\hbar\varepsilon\alpha(\frac{\partial}{\partial R}\frac{\theta}{2}\cos\varphi - \frac{\partial}{\partial R}\varphi\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\varphi), \tag{B 3b}$$

$$B_z^{(FF)}(t) = B_z^{(0)}(R) + 2\hbar\varepsilon\alpha\frac{\partial}{\partial R}\varphi\sin^2\frac{\theta}{2}. \tag{B 3c}$$

So far, in our main text, we derived the driving field which accelerates the adiabatic dynamics except for the uniform phase. Now, let us choose the regularized standard state with the adiabatic phase as

$$\Psi_0^{(reg)}(R(t)) = \begin{pmatrix} c_1(R(t)) \\ c_2(R(t)) \end{pmatrix} e^{-\frac{i}{\hbar} \int_0^t E(R(t))dt} e^{i\xi(t)}, \tag{B 4}$$

where $\xi(t)$ is the adiabatic phase given by

$$\begin{aligned} \xi(t) &= i \int_0^t dt' (c_1^* \frac{\partial c_1}{\partial t} + c_2^* \frac{\partial c_2}{\partial t}), \tag{B 5} \\ &= i\varepsilon \int_0^t dt' (c_1^* \frac{\partial c_1}{\partial R} + c_2^* \frac{\partial c_2}{\partial R}). \tag{B 6} \end{aligned}$$

In this case, $h_{11}$ and $h_{12}$ in the regularized Hamiltonian corresponding to equation (5.14) are represented as

$$\begin{aligned} h_{11} &= i\hbar(c_1^* \frac{\partial c_1}{\partial R} + c_2 \frac{\partial c_2^*}{\partial R}) + L(|c_1|^2 - |c_2|^2), \tag{B 7a} \\ h_{12} &= i\hbar(c_2^* \frac{\partial c_1}{\partial R} - c_1 \frac{\partial c_2^*}{\partial R}) + 2Lc_1 c_2^*, \tag{B 7b} \end{aligned}$$

where

$$L \equiv -i\hbar(c_1^* \frac{\partial c_1}{\partial R} + c_2^* \frac{\partial c_2}{\partial R}). \tag{B 8}$$

In the analogous way used in Section 5, the driving Hamiltonian is represented as equation (5.21). With the use of equations (5.21), (B 2), (B 6) and (B 7), we obtain the driving field as

$$\begin{aligned} B_x^{(FF)}(t) &= B_x^{(0)}(R) - \hbar\varepsilon\alpha(t)(\frac{\partial\theta}{\partial R}\sin\varphi + \frac{\partial\varphi}{\partial R}\sin\theta\cos\theta\cos\varphi), \quad \text{(B 9a)} \\ B_y^{(FF)}(t) &= B_y^{(0)}(R) + \hbar\varepsilon\alpha(\frac{\partial\theta}{\partial R}\cos\varphi - \frac{\partial\varphi}{\partial R}\sin\theta\cos\theta\sin\varphi), \quad \text{(B 9b)} \\ B_z^{(FF)}(t) &= B_z^{(0)}(R) + \hbar\varepsilon\alpha\frac{\partial\varphi}{\partial R}\sin^2\theta. \quad \text{(B 9c)} \end{aligned}$$

This kind of driving field was already obtained by Berry (Berry 2009), but is different from ours which accelerates the adiabatic dynamics without the adiabatic phase. In fact, we see the polar angle $\theta$ in equation (B 9) wherever $\theta/2$ appears in equation (B 3). Interestingly, in our examples of spin inversion and LZ model where no adiabatic phase appears ($: \frac{\partial\varphi}{\partial R} = 0$), both equations (B 3) and (B 9) lead to the identical driving field. In particular the $y$-component of $\mathbf{B}_{FF}$ in Landau-Majorana-Zener model (Landau 1932; Majorana 1932; Zener 1932) is available from both equations (B 3) and (B 9). The distinction between equations (B 3) and (B 9) will manifest itself in the case of the winding Landau-Zener model (Berry 1990; Nakamura & Rice 1994; Bouwmeester *et.al.* 1996), where the adiabatic phase is non-vanishing.

# References

Aharonov, Y. & Anandan, J. 1987 Phase change during a cyclic quantum evolution. *Phys. Rev. Lett.* **58**, 1593-1596.

Berry, M. V. 1984 Quantal phase factors accompanying adiabatic changes. *Proc. R. Soc. Lond.* **392**, 45-57.

Berry, M. V. 1990 Geometric amplitude factors in adiabatic quantum transitions. *Proc. R. Soc.* A **430**, 405-411.

Berry, M. V. 2009 Transitionless quantum driving. *J. Phys.* A **42**, 365303-365311.

Born, M. & Fock, V. 1928 Beweis des adiabaten satzes. *Z. Phys.* **51**, 165-180.

Bouwmeester, D., Karman, G. P., Schrama, C. A. & Woerdman, J. P. 1996 Observation of interference in transitions due to local geometric phases. *Phys. Rev.* A **53**, 985-989.

Chen, Xi., Ruschhaupt, A., Schmidt, S., Campo, A. del., Guéry-Odelin, D. & Muga, J. G. 2010 Fast optimal frictionless atom cooling in harmonic traps: shortcut to adiabaticity. *Phys. Rev. Lett.* **104**, 063002-063006.

Eigler, D. E. & Schweizer, E. K. 1990 Positioning single atoms with a scanning tunnelling microscope. *Nature* (London) **344**, 524-526.

Farhi, F., Goldstone, J., Gutmann, S., Lapan, J., Lundgren, A. & Preda, D. 2001 A quantum adiabatic evolution algorithm applied to random instances of an NP-complete problem. *Sience* **292**, 472-475.

Gustavson, T. L., Chikkatur, A. P., Leanhardt, A. E., Örlitz, A., Gupta, S., Pritchard, D. E. & Ketterle, W. 2002 Transport of Bose-Einstein condensates with optical tweezers. *Phys. Rev. Lett.* **88**, 020401-020404.

Kato, T. 1950 On the adiabatic theorem of quantum mechanics. *J. Phys. Soc. Jpn.* **5**, 435-439.

Ketterle, W. 2002 Nobel lecture: When atoms behave as waves: Bose-Einstein condensation and the atom laser. *Rev. Mod. Phys.* **74**, 1131-1151.

Landau, L. 1932 Zur Theorie der Energieubertragung II. *Phys. Sov. Union* **2**, 46-51.

Leanhardt, A. E., Chikkatur, A. P., Kielpinski, D., Shin, Y., Gustavson, T. L., Ketterle, W. & Pritchard, D. E. 2002 Propagation of Bose-Einstein condensates in a magnetic waveguide. *Phys. Rev. Lett.* **89**, 040401-040404.

Leggett, A. J. 2001 Bose-Einstein condensation in the alkali gases: Some fundamental concepts. *Rev. Mod. Phys.* **73**, 307-356.

Majorana, E. 1932 Atomi orientation campo magnetico variable. *Nuovo Cimento*, **9**, 43-50.

Masuda, S. & Nakamura, K. 2008 Fast-forward problem in quantum mechanics. *Phys. Rev.* A **78**, 062108-062117.

Masuda, S. & Nakamura, K. 2010 Fast-forward of adiabatic dynamics in quantum mechanics. *Proc. R. Soc.* A **466**, 1135-1154.

Messiah, A. 1962 Quantum mechanics. **2**. North-Holland: Amsterdam.

Muga, J. G., Chen, Xi., Ruschhaupt, A. & Guéry-Odelin, D. 2009 Frictionless dynamics of Bose-Einstein condensates under fast trap variations. *J. Phys.* B **42**, 241001-241005.

Nakamura, K. & Rice, S. A. 1994 Nonadiabatic transitions and gauge structure. *Phys. Rev.* A **49**, R2217-R2219.

Nielsen, M. A. & Chuang, I. L. 2000 Quantum computation and quantum information. Cambridge Univ. Press, Cambridge, UK.

Palao, J. P., Muga, J. G. & Sala, R. 1998 Composite absorbing potentials. *Phys Rev. Lett.* **80**, 5469-5472.

Roland, J. & Cerf, N. J. 2002 Quantum search by local adiabatic evolution. *Phys. Rev.* A **65**, 042308-042314.

Samuel, J. & Bhandari, R. 1988 General setting for Berry's phase. *Phys. Rev. Lett.* **60**, 2339-2342.

Sarandy, M. S. & Lidar, D. A. 2005 Simulation of Heisenberg XY interactions and realization of a perfect state transfer in spin chains using liquid nuclear magnetic resonance. *Phys. Rev.* A **71**, 012331-012343.

Shapere, A., & Wilczek, F. (eds) 1989 Geometric phase in physics, Singapore: World Scientific.

Thouless, D. J., Kohmoto, M., Nightingale, M. P., & Nijs, M. 1982 Quantized Hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.* **49** 405-408.

Zener, C. 1932 Non-adiabatic crossing of energy levels. *Proc. R. Soc. Lond.* A **137**, 696-702.